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Deformed quantum mechanics and q-Hermitian operators

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Abstract
Starting on the basis of the non-commutative q-differential calculus, we introduce a generalized q-deformed Schrödinger equation. It can be viewed as the quantum stochastic counterpart of a generalized classical kinetic equation, which reproduces at the equilibrium the well-known q-deformed exponential stationary distribution. In this framework, q-deformed adjoint of an operator and q-Hermitian operator properties occur in a natural way in order to satisfy the basic quantum mechanics assumptions.

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1. Introduction

In the recent past, there has been a great deal of interest in the study of quantum algebra and quantum groups in connection with several physical fields [1]. From the seminal work of Biedenharn [2] and Macfarlane [3], it was clear that the q-calculus, originally introduced in the study of the basic hypergeometric series [4–6], plays a central role in the representation of the quantum groups with a deep physical meaning and not merely a mathematical exercise. Many physical applications have been investigated on the basis of the q-deformation of the Heisenberg algebra [7–11]. In [12, 13], it was shown that a natural realization of quantum thermostatistics of q-deformed bosons and fermions can be built on the formalism of the q-calculus. In [14], a q-deformed Poisson bracket, invariant under the action of the q-symplectic group, has been derived and a classical q-deformed thermostatistics has been proposed in [15]. Furthermore, it is remarkable to observe that the q-calculus is very well suited to describe fractal and multifractal systems. As soon as the system exhibits a discrete-scale invariance, the natural tool is provided by Jackson q-derivative and q-integral, which constitute the natural generalization of the regular derivative and integral for discretely self-similar systems [16].

In the past, the study of generalized linear and nonlinear Schrödinger equations has attracted a lot of interest because many collective effects in quantum many-body models can be described by means of effective theories with a generalized one-particle Schrödinger equation [17–20]. On the other hand, it is relevant to mention that in recent years many investigations in literature have been devoted to non-Hermitian and pseudo-Hermitian quantum mechanics [21–25, 27].

In the framework of the q-Heisenberg algebra, q-deformed Schrödinger equations have been proposed [28, 29]. Although the proposed quantum dynamics is based on the noncummutative differential structure on configuration space, we believe that a fully consistent q-deformed formalism of the quantum dynamics, based on the properties of the q-calculus, has been still lacking.

In this paper, starting on a generalized classical kinetic equation reproducing as stationary distribution of the well-known q-exponential function, we study a generalization of the quantum dynamics consistently with the prescriptions of the q-differential calculus. At this scope, we introduce a q-deformed Schrödinger equation with a deformed Hamiltonian which is a non-Hermitian operator with respect to the standard (undeformed) operators properties but its dynamics satisfies the basic assumptions of the quantum mechanics under generalized operators properties, such as the definition of q-adjoint and q-Hermitian operators.

2. Noncommutative differential calculus

We shall briefly review the main features of the noncommutative differential q-calculus for real numbers. It is based on the following q-commutative relation among the operators $\hat{x}$ and $\hat{\partial}_x$:

$$\hat{\partial}_x \hat{x} = 1 + q\hat{x}\hat{\partial}_x, \tag{1}$$

with q a real and positive parameter.
A realization of the above algebra in terms of ordinary real numbers can be accomplished by the replacement [14, 30]

$$\hat{x} \to x, \tag{2}$$

$$\hat{\partial}_x \to \mathcal{D}_x^{(q)}, \tag{3}$$

where $\mathcal{D}_x^{(q)}$ is the Jackson derivative [4] defined as

$$\mathcal{D}_x^{(q)} = \frac{D_x^{(q)} - 1}{(q-1)x}, \tag{4}$$

where

$$D_x^{(q)} = q^{x\partial_x} \tag{5}$$

is the dilatation operator. Its action on an arbitrary real function $f(x)$ is given by

$$\mathcal{D}_x^{(q)} f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{6}$$

The Jackson derivative satisfies some simple properties which will be useful in the following. For instance, its action on a monomial $f(x) = x^n$ is given by

$$\mathcal{D}_x^{(q)} x^n = [n]_q x^{n-1} \tag{7}$$

and

$$\mathcal{D}_x^{(q)} x^{-n} = -\frac{[n]_q}{q^n} \frac{1}{x^{n+1}}, \tag{8}$$

where $n \geqslant 0$ and

$$[n]_q = \frac{q^n - 1}{q - 1} \tag{9}$$

are the so-called basic numbers. Moreover, we can easily verify the following q-version of the Leibnitz rule:

$$\begin{aligned}\mathcal{D}_x^{(q)}(f(x)g(x)) &= \mathcal{D}_x^{(q)} f(x)g(x) + f(qx)\mathcal{D}_x^{(q)} g(x),\\ &= \mathcal{D}_x^{(q)} f(x)g(qx) + f(x)\mathcal{D}_x^{(q)} g(x).\end{aligned} \tag{10}$$

A relevant role in the q-algebra, as developed by Jackson, is given by the *basic* binomial series defined by

$$\begin{aligned}(x+y)^{(n)} &= (x+y)(x+qy)(x+q^2y)\cdots(x+q^{n-1}y)\\ &\equiv \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q q^{r(r-1)/2} x^{n-r} y^r,\end{aligned} \tag{11}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!} \tag{12}$$

is known as the q-binomial coefficient which reduces to the ordinary binomial coefficient in the $q \to 1$ limit [6]. We should remark that equation (12) holds for $0 \leqslant r \leqslant n$, while it is assumed to vanish otherwise and we have defined $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. Remarkably, a q-analogue of the Taylor expansion has been introduced in [4] by means of a *basic* binomial (11) as

$$f(x) = f(a) + \frac{(x-a)^{(1)}}{[1]!} \mathcal{D}_x^{(q)} f(x)\Big|_{x=a} + \frac{(x-a)^{(2)}}{[2]!} \mathcal{D}_x^{(q)2} f(x)\Big|_{x=a} + \cdots, \tag{13}$$

where $\mathcal{D}_x^{(q)2} \equiv \mathcal{D}_x^{(q)}\mathcal{D}_x^{(q)}$ and so on.
Consistently with the q-calculus, we also introduce the *basic* integration

$$\int_0^{\lambda_0} f(x)\,\mathrm{d}_q x = \sum_{n=0}^{\infty} \Delta_q \lambda_n f(\lambda_n), \tag{14}$$

where $\Delta_q \lambda_n = \lambda_n - \lambda_{n+1}$ and $\lambda_n = \lambda_0 q^n$ for $0 < q < 1$ whilst $\Delta_q \lambda_n = \lambda_{n-1} - \lambda_n$ and $\lambda_n = \lambda_0 q^{-n-1}$ for $q > 1$ [5, 6, 15, 16]. Clearly, equation (14) is reminiscent of the Riemann quadrature formula performed now in a q-nonuniform hierarchical lattice with a variable step $\Delta_q \lambda_n$. It is trivial to verify that

$$\mathcal{D}_x^{(q)} \int_0^x f(y)\,\mathrm{d}_q y = f(x), \tag{15}$$

for any $q > 0$.

Let us now introduce the following q-deformed exponential function defined by the series:

$$\mathrm{E}_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = 1 + x + \frac{x^2}{[2]_q!} + \frac{x^3}{[3]_q!} + \cdots, \tag{16}$$

which will play an important role in the framework we are introducing. The function (16) defines the *basic* exponential, well known in the literature since a long-time ago, originally introduced in the study of basic hypergeometric series [5, 6]. In this context, let us observe that definition (16) is fully consistent with its Taylor expansion, as given by equation (13).

The *basic* exponential is a monotonically increasing function, $\mathrm{dE}_q(x)/\mathrm{d}x > 0$, convex, $\mathrm{d}^2\mathrm{E}_q(x)/\mathrm{d}x^2 > 0$, with $\mathrm{E}_q(0) = 1$ and reducing to the ordinary exponential in the $q \to 1$ limit: $\mathrm{E}_1(x) \equiv \exp(x)$. An important property satisfied by the q-exponential can be written formally as [6]

$$\mathrm{E}_q(x+y) = \mathrm{E}_q(x)\mathrm{E}_{q^{-1}}(y), \tag{17}$$

where the left-hand side of equation (17) must be considered by means of its series expansion in terms of *basic* binomials,

$$\mathrm{E}_q(x+y) = \sum_{k=0}^{\infty} \frac{(x+y)^{(k)}}{[k]!}. \tag{18}$$

By observing that $(x-x)^{(k)} = 0$ for any $k > 0$, since $(x-x)^{(0)} = 1$, from equation (17) we can see that [15]

$$\mathrm{E}_q(x)\mathrm{E}_{q^{-1}}(-x) = 1. \tag{19}$$

The above property will be crucial in the following introduction to a consistent q-deformed quantum mechanics.

Among many properties, it is important to recall the following relation [6]:

$$\mathcal{D}_x^{(q)}\mathrm{E}_q(ax) = a\mathrm{E}_q(ax), \tag{20}$$

and its dual

$$\int_0^x \mathrm{E}_q(ay)\,\mathrm{d}_q y = \frac{1}{a}[\mathrm{E}_q(ax) - 1]. \tag{21}$$

Finally, it should be pointed out that equations (20) and (21) are two important properties of the *basic* exponential which turns out to be not true if we employ the ordinary derivative or integral.

3. Classical q-deformed kinetic equation

Starting from the realization of the q-algebra, defined in equations (2)–(3), we can write for a homogeneous system the following q-deformed Fokker–Planck equation [31]:

$$\frac{\partial f_q(x,t)}{\partial t} = \mathcal{D}_x^{(q)}\big[-J_1^{(q)}(x) + J_2^{(q)}\mathcal{D}_x^{(q)}\big] f_q(x,t), \tag{22}$$

where $J_1^{(q)}(x)$ and $J_2^{(q)}$ are the drift and diffusion coefficients, respectively.

The above equation has stationary solution $f_{\mathrm{st}}^{(q)}(x)$ that can be written as

$$f_{\mathrm{st}}^{(q)}(x) = N_q\mathrm{E}_q[-\Phi_q(x)], \tag{23}$$

where N_q is a normalization constant, $E_q[x]$ is the q-deformed exponential function defined in equation (16) and we have defined[1]

$$\Phi_q(x) = -\frac{1}{J_2^{(q)}}\int_0^x J_1^{(q)}(y)\,\mathrm{d}_q y. \tag{24}$$

If we postulate a generalized Brownian motion in a q-deformed classical dynamics by mean the following definition of the drift and diffusion coefficients:

$$J_1^{(q)}(x) = -\gamma x\left(q D_x^{(q)} + 1\right), \qquad J_2^{(q)} = \gamma/\alpha, \tag{25}$$

where γ is the friction constant, α is a constant depending on the system and $D_x^{(q)}$ is the dilatation operator (5), the stationary solution $f_{\mathrm{st}}^{(q)}(x)$ of the above Fokker–Planck equation can be obtained as solution of the following stationary q-differential equation:

$$\mathcal{D}_x^{(q)} f_{\mathrm{st}}^{(q)}(x) = -\alpha x\big[q f_{\mathrm{st}}^{(q)}(qx) + f_{\mathrm{st}}^{(q)}(x)\big]. \tag{26}$$

It easy to show that the solution of the above equation can be written as

$$f_{\mathrm{st}}^{(q)}(x) = N_q\mathrm{E}_q[-\alpha x^2]. \tag{27}$$

[1] In the following, for simplicity, we limit ourselves to consider the drift coefficient as a monomial function of x.

4. q-deformed Schrödinger equation

We are now able to derive a q-deformed Schrödinger equation by means of a stochastic quantization method [32].

Starting from the following transformation of the probability density:

$$f_q(x,t) = \mathrm{E}_q[-\frac{\Phi_q(x)}{2}]\psi_q(x,t), \tag{28}$$

where $\Phi_q(x)$ is the function defined in equation (24), the q-deformed Fokker–Planck equation (22) can be written as

$$\frac{\partial \psi_q(x,t)}{\partial t} = J_2^{(q)} \mathcal{D}_x^{(q)^2} \psi_q(x,t) - V_q(x)\psi_q(x,t), \tag{29}$$

where

$$V_q(x) = \left\{ \frac{1}{2}\mathcal{D}_x^{(q)} J_1^{(q)}(x) + \frac{\left[J_1^{(q)}(x)\right]^2}{4J_2^{(q)}} \right\}. \tag{30}$$

The above equation has the same structure of the time-dependent Schrödinger equation. In fact, the stochastic quantization of equation (22) can be realized with the transformations

$$t \to \frac{t}{-\mathrm{i}\hbar}, \tag{31}$$

$$J_2^{(q)} \to \frac{\hbar^2}{2m}, \tag{32}$$

getting the q-generalized Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial \psi_q(x,t)}{\partial t} = H_q \psi_q(x,t), \tag{33}$$

where

$$H_q = -\frac{\hbar^2}{2m}\mathcal{D}_x^{(q)^2} + V_q(x) \tag{34}$$

is the q-deformed Hamiltonian. Let us note that the Hamiltonian (34) is a not-Hermitian operator with respect to the standard definition based on the ordinary (undeformed) scalar product of square-integrable functions [9, 14]. In the following section, we will see as this aspect can be overridden by means the introduction of a q-deformed scalar product and generalized properties of operators inspired to the q-calculus.

The above equation admits factorized solution $\psi_q(x,t) = \phi(t)\varphi_q(x)$, where $\phi(t)$ satisfies to the equation

$$\mathrm{i}\hbar \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = E\phi(t), \tag{35}$$

with the standard (undeformed) solution

$$\phi(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}Et\right), \tag{36}$$

while $\varphi_q(x)$ is the solution of time-independent q-Schrödinger equation

$$H_q\varphi_q(x) = E\varphi_q(x). \tag{37}$$

In one-dimensional case, for a free particle ($V_q = 0$) described by the wavefunction $\varphi_q^f(x)$, equation (37) becomes

$$\mathcal{D}_x^{(q)^2}\varphi_q^f(x) + k^2\varphi_q^f(x) = 0, \tag{38}$$

where $k = \sqrt{2mE/\hbar^2}$. The solution of the previous equation can be written as

$$\varphi_q^f(x) = N\mathrm{E}_q(\mathrm{i}kx). \tag{39}$$

The above equation generalizes the plane wavefunction in the framework of the q-calculus.

5. q-deformed products and q-Hermitian operators

In order to develop a consistent deformed quantum dynamics, we have to generalize the products between functions and properties of the operators in the framework of the q-calculus.

Let us start on the basis of equation (19), which implies

$$\mathrm{E}_q(\mathrm{i}x)(\mathrm{E}_{q^{-1}}(\mathrm{i}x))^\star = 1, \tag{40}$$

$$\mathrm{E}_q(\mathrm{i}x)^\star = (\mathrm{E}_{q^{-1}}(\mathrm{i}x))^{-1}, \tag{41}$$

and in terms of the q-plane wave (39)

$$\varphi_{q^{-1}}^f(x)^\star \varphi_q^f(x) = N^2. \tag{42}$$

Inspired to the above equation, it appears natural to introduce the complex q-conjugation of a function as

$$\psi_q^\dagger(x) = \psi_{q^{-1}}^\star(x), \tag{43}$$

and, consequently, the probability density of a single particle in a finite space as

$$\rho_q(x,t) = |\psi_q(x,t)|_q^2 = \psi_q^\dagger(x,t)\psi_q(x,t) \equiv \psi_{q^{-1}}^*(x,t)\psi_q(x,t). \tag{44}$$

Thus, the wavefunctions must be q-square-integrable functions of configuration space, that is to say the functions $\psi_q(x)$ such that the integral

$$\int |\psi_q(x)|_q^2 \, \mathrm{d}_q x \tag{45}$$

converges.

The function space defined above is a linear space. If ψ_q and φ_q are two q-square-integrable functions, any linear combinations $\alpha\psi_q + \beta\varphi_q$, where α and β are arbitrarily chosen complex numbers, are also q-square-integrable functions.

Following this line, it is possible to define a q-scalar product of the function ψ by the function φ as

$$\langle\varphi, \psi\rangle_q = \int \varphi_q^\dagger(x)\psi_q(x)\,\mathrm{d}_q x \equiv \int \varphi_{q^{-1}}^\star(x)\psi_q(x)\,\mathrm{d}_q x. \tag{46}$$

This is linear with respect to ψ, the norm of a function ψ_q is a real, non-negative number, $\langle\psi, \psi\rangle_q \geqslant 0$ and

$$\langle\psi, \varphi\rangle_q = \langle\varphi, \psi\rangle_q^\dagger. \tag{47}$$

Analogously to the undeformed case, it is easy to see that from the above properties of the q-scalar product follows the q-Schwarz inequality

$$|\langle\varphi, \psi\rangle_q|_q^2 \leqslant \langle\varphi, \varphi\rangle_q \langle\psi, \psi\rangle_q. \tag{48}$$

Consistently with the above definitions, the q-adjoint of an operator A_q is defined by means of the relation

$$\left\langle\psi, A_q^\dagger\varphi\right\rangle_q = \langle\varphi, A_q\psi\rangle_q^\dagger, \tag{49}$$

and, by definition, a linear operator is q-Hermitian if it is its own q-adjoint. More explicitly, an operator A_q is q-Hermitian if for any two states φ_q and ψ_q we have

$$\langle\varphi, A_q\psi\rangle_q = \langle A_q\varphi, \psi\rangle_q. \tag{50}$$

First of all, the above properties are crucial to have a consistent conservation in time of the probability densities, defined in equation (44). In fact, by taking the complex q-conjugation of equation (33), summing and integrating term by term the two equations, we get

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\int \psi_q^\dagger\psi_q\,\mathrm{d}_q x = \int \left[\psi_{q^{-1}}^\star(H_q\psi_q) - (H_{q^{-1}}\psi_{q^{-1}}^\star)\psi_q\right]\mathrm{d}_q x = 0, \tag{51}$$

where the last equivalence follows from the fact that the operator Hamiltonian is q-Hermitian. In this context, it is relevant to observe that it is possible to verify the above property by using the time-spatial factorization solution $\psi_q(x,t) = \phi(t)\varphi_q(x)$ of the q-Schrödinger equation. In fact, we have

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\int \psi_q^\dagger\psi_q\,\mathrm{d}_q x = \int \phi^\star\phi\left[\varphi_{q^{-1}}^\star(H_q\varphi_q) - (H_{q^{-1}}\varphi_{q^{-1}}^\star)\varphi_q\right]\mathrm{d}_q x. \tag{52}$$

From the stationary Schrödinger equation (37) and its complex q-conjugation we have directly

$$\varphi_{q^{-1}}^\star(H_q\varphi_q) = (H_{q^{-1}}\varphi_{q^{-1}}^\star)\varphi_q, \tag{53}$$

and the terms in the square bracket of equation (52) go to zero.

6. Observables in q-deformed quantum mechanics

On the basis of the above properties, we have the recipe to generalize the definition of observables in the framework of q-deformed theory by postulating that:

- with the dynamical variable $A(x,p)$ associate the linear operator $A_q\big(x, -\mathrm{i}\hbar\mathcal{D}_x^{(q)}\big)$;
- the mean value of this dynamical variable, when the system is in the dynamical (normalized) state ψ_q, is

$$\langle A\rangle_q = \int \psi_q^\dagger A_q\psi_q\,\mathrm{d}_q x \equiv \int \psi_{q^{-1}}^\star A_q\psi_q\,\mathrm{d}_q x. \tag{54}$$

Observables are real quantities, hence the expectation value (54) must be real for any state ψ_q,

$$\int \psi_q^\dagger A_q\psi_q\,\mathrm{d}_q x = \int (A_q\psi_q)^\dagger\psi_q\,\mathrm{d}_q x, \tag{55}$$

therefore, on the basis of equation (50), observables must be represented by q-Hermitian operators.

If we require there is a state ψ_q for which the result of measuring the observable A is unique, in other words that the fluctuations

$$(\Delta A_q)^2 = \int \psi_q^\dagger(A_q - \langle A\rangle_q)^2\psi_q\,\mathrm{d}_q x \tag{56}$$

must vanish, we obtain the following q-eigenvalue equation of a q-Hermitian operator A_q with eigenvalue a:

$$A_q\varphi_q = a\varphi_q. \tag{57}$$

As a consequence, the eigenvalues of a q-Hermitian operator are real because $\langle A\rangle_q$ is real for any state; in particular, for an eigenstate with the eigenvalue a for which $\langle A\rangle_q = a$.

Furthermore, as in the undeformed case, two eigenfunctions $\psi_{q,1}$ and $\psi_{q,2}$ of the q-Hermitian operator A_q, corresponding to different eigenvalues a_1 and a_2, are orthogonal. We can always normalize the eigenfunction; therefore, we can choose all the eigenvalues of a q-Hermitian operator orthonormal, i.e.,

$$\int \psi_{q,n}^{\dagger} \psi_{q,m} \, \mathrm{d}_q x = \delta_{n,m}. \tag{58}$$

Consequently, two eigenfunctions $\psi_{q,1}$ and $\psi_{q,2}$ belonging to different eigenvalues are linearly independent.

It is easy to see that, adapting step by step the undeformed case to the introduced q-deformed framework, the totality of the linearly independent eigenfunctions $\{\psi_{q,n}\}$ of q-Hermitian operator A_q form a complete (orthonormal) set in the space of the previously introduced q-square-integrable functions. In other words, if ψ_q is any state of a system, then it can be expanded in terms of the eigenfunctions (with a discrete spectrum) of the corresponding q-Hermitian operator A_q associate with the observable

$$\psi_q = \sum_n c_{q,n} \psi_{q,n}, \tag{59}$$

where

$$c_{q,n} = \int \psi_{q,n}^{\dagger} \psi_q \, \mathrm{d}_q x. \tag{60}$$

The above expansion allows us, as usual, to write the expectation value of A_q in the normed state ψ_q as

$$\langle A \rangle_q = \int \psi_q^{\dagger} A_q \psi_q \, \mathrm{d}_q x = \sum_n |c_{q,n}|_q^2 a_n, \tag{61}$$

where $\{a_n\}$ are the set of eigenvalues (assumed, for simplicity, discrete and non-degenerate) and the normalization condition of the wavefunction can be written in the form

$$\sum_n |c_{q,n}|_q^2 = 1. \tag{62}$$

7. Conclusions

On the basis of the stochastic quantization procedure and on the q-differential calculus, we have obtained a generalized linear Schrödinger equation which involves a q-deformed Hamiltonian that is non-Hermitian with respect to the standard (undeformed) definition. However, under an appropriate generalization of the operators properties and the introduction of a q-deformed scalar product in the space of q-square-integral wavefunctions, such equation of motion satisfies the basic quantum mechanics assumptions.

Although a complete physical and mathematical description of the introduced quantum-dynamical equations lies out the scope of this paper, we think that the results derived here appear to provide a deeper insight into a full consistent q-deformed quantum mechanics in the framework of the q-calculus and may be a relevant starting point for future investigations.

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References

[1] Baxter R 1982 *Exact Solved Models in Statistical Mechanics* (New York: Academic)
[2] Biedenharn L 1989 *J. Phys. A: Math. Gen.* **22** L873
[3] Macfarlane A 1989 *J. Phys. A: Math. Gen.* **22** 4581
[4] Jackson F H 1909 *Am. J. Math.* **38** 26
Jackson F H 1909 *Mess. Math.* **38** 57
[5] Gasper G and Rahman M 1990 Basic hypergeometric series *Encyclopedia of Mathematics and its Applications* (Cambridge: Cambridge Univeristy Press)
[6] Exton H 1983 *q-Hypergeometric Functions and Applications* (Chichester: Ellis Horwood)
[7] Wess J and Zumino B 1990 *Nucl. Phys.* B **18** 302
[8] Celeghini E *et al* 1995 *Ann. Phys.* **241** 50
[9] Cerchiai B L, Hinterding R, Madore J and Wess J 1999 *Eur. Phys. J.* C **8** 547
Cerchiai B L, Hinterding R, Madore J and Wess J 1999 *Eur. Phys. J.* C **8** 533
[10] Bardek V and Meljanac S 2000 *Eur. Phys. J.* C **17** 539
[11] Finkelstein R 1996 *J. Math. Phys.* **37** 3645
Finkelstein R 1996 *J. Math. Phys.* **37** 2628
[12] Lavagno A and Swamy N P 2000 *Phys. Rev.* E **61** 1218
Lavagno A and Swamy N P 2002 *Phys. Rev.* E **65** 036101
[13] Lavagno A 2002 *Phys. Lett.* A **301** 13
Lavagno A 2002 *Physica* A **305** 238
[14] Lavagno A, Scarfone A M and Swamy N P 2006 *Eur. Phys. J.* C **47** 253
[15] Lavagno A, Scarfone A M and Swamy N P 2007 *J. Phys. A: Math. Theor.* **40** 8635
[16] Erzan A and Eckmann J-P 1997 *Phys. Rev. Lett.* **78** 3245
[17] Ginsburg V and Pitayevski L 1958 *Sov. Phys.—JETP* **7** 858
[18] Gross E P 1963 *J. Math. Phys.* **4** 195
[19] Doebner H D and Goldin G 1992 *Phys. Lett.* A **162** 397
[20] Sutherland B 1985 *Lectures Notes in Physics* vol 242 (Berlin: Springer)
[21] Mostafazadeh A 2002 *J. Math. Phys.* **43** 3944
[22] Bender C M, Brody D C and Jones H F 2002 *Phys. Rev. Lett.* **89** 270401
[23] Mostafazadeh A 2003 *J. Phys. A: Math. Gen.* **33** 7081
[24] Bender C M, Brody D C and Jones H F 2004 *Phys. Rev. Lett.* **93** 251601
[25] Jones H F 2005 *J. Phys. A: Math. Gen.* **38** 1741
[26] Jones H F and Mateo J 2006 *Phys. Rev.* D **73** 085002
[27] Bender C M, Brody D C, Jones H F and Meister B K 2007 *Phys. Rev. Lett.* **98** 040403
[28] Zhang J-zu 1998 *Phys. Lett.* B **440** 66
Zhang J-zu 2000 *Phys. Lett.* B **477** 361
[29] Micu M 1999 *J. Phys. A: Math. Gen.* **32** 7765
[30] Ubriaco M R 1992 *J. Phys. A: Math. Gen.* **25** 169
[31] Lavagno A, Scarfone A M and Swamy N P 2006 *Eur. Phys. J.* B **50** 351
[32] Risken H 1989 *The Fokker–Planck Equation* (Berlin: Springer)